A Banach space with a symmetric basis which is of weak cotype 2 but not of cotype 2

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Abstract

We prove that the symmetric convexified Tsirelson space is of weak cotype 2 but not of cotype 2.

Introduction

Weak type 2 and weak cotype 2 spaces were originally introduced and investigated by V.D. Milman and G. Pisier in [11] and weak Hilbert spaces by Pisier in [13]. A further detailed investigation can be found in Pisier's book [14]. The first example of a weak Hilbert space which is not isomorphic to a Hilbert space is the 2-convexified Tsirelson space (called the convexified Tsirelson space in this paper). This follows from the results of W.B. Johnson in [5]. For a detailed study of the original Tsirelson space we refer to [3].

Let X be a Banach space with a symmetric basis. It was proved in [14] that if X is a weak Hilbert space, then it is isomorphic to a Hilbert space and this has lead to the belief that if X is just of weak cotype 2, then it is of cotype 2. However, this turns out not necessarily to be the case. The main result of this paper states that the symmetric convexified Tsirelson space is of weak cotype 2 but not of cotype 2.

We now wish to discuss the arrangement of this paper in greater detail.

In Section 1 we give some basic facts on properties related to weak type 2 and weak cotype 2 while Section 2 is devoted to a review of some results on the convexified Tsirelson space which

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we need for our main result. Most of these results are stated without proofs since they can be proved in a similar manner as the corresponding results for the original Tsirelson space.

In Section 3 we make the construction of the symmetric convexified Tsirelson space, investigate its basic properties and prove our main result stated above.

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1 Notation and Preliminaries

In this paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [9], [10] and [16]. B_X shall always denote the closed unit ball of the Banach space X and if X and Y are Banach spaces, then $B(X,Y)$ ($B(X) = B(X,X)$) denotes the space of all bounded linear operators from X to Y.

We let (g_n) denote a sequence of independent standard Gaussian variables on a fixed probability space $(\Omega, \mathcal{S}, \mu)$ and recall that a Banach space X is said to be of type p, $1 \leq p \leq 2$, (respectively cotype p, $2 \leq p < \infty$) if there is a constant $K \geq 1$ so that for all finite sets $\{x_1, x_2, \ldots, x_n\} \subseteq X$ we have

$$\Big(\int \Big\|\sum_{j=1}^{n} g_j(t)x_j\Big\|^p d\mu(t)\Big)^{\frac{1}{p}} \leq K\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{1}{p}} \tag{1.1}$$

(respectively

$$K\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{\frac{1}{p}} \leq \Big(\int \Big\|\sum_{j=1}^{n} g_j(t)x_j\Big\|^p d\mu(t)\Big)^{\frac{1}{p}}). \tag{1.2}$$

The smallest constant K which can be used in (1.1) (respectively (1.2)) is denoted by $K^p(X)$ (respectively $K_p(X)$).

If L is a Banach lattice and $1 \leq p < \infty$, then L is said to be p-convex (respectively p-concave)

if there is a constant $C \geq 1$ so that for all finite sets $\{x_1, x_2, \ldots, x_n\} \subseteq L$ we have

$$\|(\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}}\| \leq C(\sum_{j=1}^{n} \|x_j\|^p)^{\frac{1}{p}} \tag{1.3}$$

(respectively

$$(\sum_{j=1}^{n} \|x_j\|^p)^{\frac{1}{p}} \leq C\|(\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}}\|). \tag{1.4}$$

The smallest constant C which can be used in (1.3) (respectively (1.4)) is denoted by $C^p(L)$ (respectively $C_p(L)$).

It follows from [10, 1.d.6 (i)] that if L is of finite concavity (equivalently of finite cotype), then there is a constant $K \geq 1$ so that

$$\frac{1}{K}\|(\sum_{j=1}^{n} |x_j|^2)^{\frac{1}{2}}\| \leq \Big(\int \|\sum_{j=1}^{n} g_j(t)x_j\|^2 d\mu(t)\Big)^{\frac{1}{2}} \leq K\|(\sum_{j=1}^{n} |x_j|^2)^{\frac{1}{2}}\| \tag{1.5}$$

A Banach space X is said to be of weak type 2 if there is a constant C and a δ, $0 < \delta < 1$, so that whenever $E \subseteq X$ is a subspace, $n \in \mathbb{N}$ and $T \in B(E, \ell_2^n)$, then there is an orthogonal projection P on ℓ_2^n of rank larger than δn and an operator $S \in B(X, \ell_2^n)$ with $Sx = PTx$ for all $x \in E$ and $\|S\| \leq C\|T\|$.

Similarly X is called a weak cotype 2 if there is a constant C and a δ, $0 < \delta < 1$, so that whenever $E \subseteq X$ is a finite dimensional subspace, then there is a subspace $F \subseteq E$ so that $\dim F \geq \delta \dim E$ and $d(F, \ell_2^{\dim F}) \leq C$.

Our definitions of weak type 2 and weak cotype 2 space are not the original ones, but are chosen out of the many equivalent characterizations given by Pisier [14].

A weak Hilbert space is a space which is both of weak type 2 and weak cotype 2.

If A is a set we let $|A|$ denote the cardinality of A.

Definition 1.1 *If (x_n) and (y_n) are sequences in a Banach space X, we say that (x_n) is dominated by (y_n) if there is a constant $K > 0$ so that for all finitely non-zero sequences of scalars (a_n) we have*

$$\|\sum_n a_n x_n\| \leq K\|\sum_n a_n y_n\|.$$

We will need some information about property (H) and related properties.

Definition 1.2 *A Banach space X has property (H_2) if there is a function $C(\cdot,\cdot)$ so that for every $0<\delta<1$ and for every normalized λ-unconditional basic sequence $(x_i)_{i=1}^n$ in X there is a subset $F\subseteq\mathbb{N}$ such that $|F|\geq\delta n$ and $(x_i)_{i\in F}$ is $C(\lambda,\delta)$-equivalent to the unit vectors basis of $\ell_2^{|F|}$. If we only have that $(x_i)_{i\in F}$ is $C(\lambda,\delta)$ dominated by the unit vector basis of $\ell_2^{|F|}$, we say that X has property upper (H_2). Similarly, we define property lower (H_2).*

Definition 1.3 *A Banach space X is said to have property (H) if there is a function $f(\cdot)$ so that for every normalized λ-unconditional basic sequence $(x_i)_{i=1}^n$ in X, we have*

$$\frac{1}{f(\lambda)}n^{1/2}\leq\|\sum_{i=1}^n x_i\|\leq f(\lambda)n^{1/2}.$$

Similarly, we can define property upper (H) and property lower (H).

The following is clear.

Proposition 1.4 *Property upper (resp. lower) (H_2) implies upper (resp. lower) (H).*

We will see later that the converses of Proposition 1.4 fail.

The next result shows that any percentage of the basis will work in the definition of (H_2). The proof follows from the argument of Pisier [14, Proposition 12.4, page 193].

Lemma 1.5 *For a Banach space X, the following are equivalent:*

(1) X has property upper (resp. lower) (H_2).

(2) There exists one $0<\delta<1$ satisfying the conclusion of property upper (resp. lower) (H_2).

The corresponding result for property (H) is in [3, Proposition Ae1, page 14].

Lemma 1.6 *For a Banach space X, the following are equivalent:*

(1) X has property upper (resp. lower) (H).

(2) There is a $0<\delta<1$ so that for every λ-unconditional basic sequence $(x_i)_{i=1}^n$ in X there is a subset $F\subset\{1,2,\cdots,n\}$ with $|F|\geq\delta n$ and $(x_i)_{i\in F}$ has property upper (resp. lower) (H).

The next theorem is due to Pisier [14, Proposition 12.4].

Proposition 1.7 *Every weak Hilbert space has property (H_2).*

We also have from Pisier [14, Proposition 10.8, page 160 and Proposition 11.9, page 174]:

Proposition 1.8 *The following implications hold for a Banach space X:*

(1) Weak cotype 2 implies property lower (H).

(2) Weak type 2 implies property upper (H).

The converses of Proposition 1.8 are open questions. However, for Banach lattices it is known that property (H), property (H_2) and being a weak Hilbert space are all equivalent. This is a result of Nielsen and Tomczak-Jaegermann [12].

2 Convexified Tsirelson Space

Since there is only a "partial theory" developed for the convexified Tsirelson space T^2, we will review what we need here.

Notation 2.1 *If E, F are sets of natural numbers, we write $E < F$ if for every $n \in E$ and every $m \in F$, $n < m$. If $E = \{k\}$, we just write $k < F$ for $E < F$.*

Definition 2.2 *We define the convexified Tsirelson space T^2 as the set of vectors $x = \sum_n a_n t_n$ for which the recursively defined norm below is finite.*

$$\|x\|_{T^2} = \mathit{max}\{\mathit{sup}|a_n|, 2^{-1/2}\mathit{sup}\left(\sum_{j=1}^{k} \|E_j x\|^2\right)^{1/2}\}, \tag{2.1}$$

where the second "sup" is taken over all choices

$$k \leq E_1 < E_2 < \cdots < E_k,$$

and $Ex = \sum_{n \in E} a_n t_n$.

We will now list the known results for this space (which we will need) and where they can be found. The first result can be found in [3] and [14].

Proposition 2.3 *The unit vectors (t_n) form a 1-unconditional basis for T^2. The space T^2 is of type 2 and weak cotype 2 but does not contain a Hilbert space.*

Next we need to see which subsequences of the unit vector basis of T^2 are equivalent to the original basis. To do this we need:

Notation 2.4 *The fast growing hierarchy from logic is a family of functions on $\mathbb{N}$ given by: $g_o(n) = n + 1$, and for $i \geq 0$, $g_{i+1}(n) = g_i^{(n)}(n)$, where for any function f, $f^{(n)}$ is the n-fold iteration of f. We also set exp$_0(n) = n$ and for $i, n \geq 1$,*

$$\mathit{exp}_i(n) = 2^{\mathit{exp}_{i-1}(n)}.$$

Finally we let log$_0(n) = n$, and for n large enough so that log$_{i-1}(n) > 0$, let

$$\mathit{log}_i(n) = \mathit{log}(\mathit{log}_{i-1}(n)).$$

The next result is due to Bellenot [1]. He does this result in the original Tsirelson's space T, but the proof works perfectly well in T^2.

Proposition 2.5 *A subsequence (t_{k_n}) of (t_n) is equivalent to (t_n) if and only if there is a natural number i so that $k_n \leq g_i(n)$, for all large n. Moreover, (t_{k_n}) always 1-dominates (t_n) and there is a constant $K \geq 1$ so that the equivalence constant is K^i for the case $g_i(n)$.*

One important consequence is (see Pisier [14] or Casazza and Shura [3]).

Proposition 2.6 *Every $g_i(n)$-dimensional subspace of span $(t_j)_{j\geq n}$ is K^i-isomorphic to a Hilbert space and K^i-complemented in T^2.*

If X is a weak Hilbert space with an unconditional basis, then it follows from [12] that the conclusion of Proposition 2.6 remains true after a suitable permutation of the basis.

The next result comes from [3, Theorem IV.b.3, page 39]. The theorem there is proved for the regular Tsirelson space but the techniques easily adapt to convexified space.

Proposition 2.7 *Every n-dimensional subspace of T^2 is K^ilog$_i(n)$ isomorphic to ℓ_2^n.*

We need one more result on convexified Tsirelson.

Proposition 2.8 *If $x = \sum_j a_j t_j \in T^2$, then for all $n \in \mathbb{N}$,*

$$\|\sum_j a_j t_{nj}\|_{T^2} \leq 2K^i(\mathit{log}_i n)\|x\|_{T^2}.$$

Proof: By Proposition 2.5 and Proposition 2.7 we have

$$\|\sum_j a_j t_{nj}\|_{T^2} \leq \|\sum_{j=1}^{n} a_j t_{nj}\|_{T^2} + \|\sum_{j=n+1}^{\infty} a_j t_{nj}\|_{T^2} \;\leq\; \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \|\sum_{j=n+1}^{\infty} a_j t_j\|_{T^2}$$
$$\leq K^i(\log_i n)\|\sum_{j=1}^{n} a_j t_j\| + K\|\sum_{j=n+1}^{\infty} a_j t_j\|_{T^2} \;\leq\; 2K^i(\log_i n)\|x\|.$$

□

3 Symmetric Convexified Tsirelson Space

There is almost no existing theory for the symmetric convexified Tsirelson space. But there is a theory for the symmetric Tsirelson space. We will list the results we need on this topic. They can be found in Casazza and Shura [3, Chapter X.E].

Notation 3.1 *For T^2 or $(T^2)^*$ we will work with the non-decreasing rearrangement operator D. That is, if $x = \sum_n a_n t_n$ then $Dx = \sum_n a_n^* t_n$ where (a_n^*) is the non-decreasing re-arrangement of the non-zero $a_n's$ where by non-decreasing we mean the absolute values are non-decreasing.*

The construction of Chapter VIII of [3, Chapters VIII and X.B] shows

Proposition 3.2 *Let Π denote the group of all permutations of $\mathbb{N}$. There is a constant $K \geq 1$ so that for any $x = \sum_n a_n t_n^* \in (T^2)^*$ we have*

$$\|x\|_{s^*} =: sup_{\sigma\in\Pi}\|\sum_n a_{\sigma(n)} t_n^*\| \leq K\|Dx\| \leq K sup_{\sigma\in\Pi}\|\sum_n a_{\sigma(n)} t_n^*\|. \tag{3.1}$$

We will define the *dual space of the symmetric convexified Tsirelson space* first because it is natural in terms of the above.

Definition 3.3 *We let $S[(T^2)^*]$ be the family of all vectors for which $\|x\|_{s^*}$ is finite. Then this is a Banach space with a natural symmetric basis, denoted (t_n^{s*}), called the dual space of the symmetric convexified Tsirelson space.*

To define the *the symmetric convexified Tsirelson space* we need a result kindly communicated to us by N.J. Kalton.

Let X be a Banach sequence space. Define the permutation operators $S_\sigma(\xi) = (\xi_{\sigma(n)})_{n=1}^\infty$ for $\sigma \in \Pi$ and let L_k^j to be the linear map such that $L_k^j(e_n) = e_{kn+j}$ for all $n \in \mathbb{N}$. Finally we let c_{00} denote the spaces af real sequences which are eventually 0.

Theorem 3.4 *Suppose X is a Banach sequence space which is p-convex and q-concave where $1 < p < q < \infty$. Suppose $\max_{0 \le j < k} \|L_k^j\| \le Ck^a$ where $a + p^{-1} < 1$. Then*

$$\|\xi\|_{X_{inf}} = \inf_{\sigma \in \Pi} \|S_\sigma \xi\|_X,\ x \in c_{00}$$

defines a quasi-norm on c_{00} which is equivalent to a norm. The dual of X_{inf} is X^*_{sup} where

$$\|\xi\|_{X^*_{sup}} = \sup_{\sigma \in \Pi} \|S_\sigma \xi\|_{X^*}.$$

Proof: Let us start by supposing $x_1, \cdots, x_k \in c_{00}$ are disjointly supported and that $\sigma_1, \cdots, \sigma_k \in \Pi$. Then

$$\begin{aligned} \|x_1 + \cdots + x_k\|_{X_{inf}} &\le \|\sum_{j=1}^k L_k^{j-1} S_{\sigma_j} x_j\|_X \\ &\le (\sum_{j=1}^k \|L_k^{j-1} S_{\sigma_j} x_j\|_X^p)^{\frac{1}{p}} \\ &\le Ck^a (\sum_{j=1}^k \|S_{\sigma_j} x_j\|_X^p)^{\frac{1}{p}}. \end{aligned}$$

Now taking an infimum over σ_j gives

$$\|x_1 + \cdots + x_k\|_{X_{inf}} \le Ck^a (\sum_{j=1}^k \|x_j\|_{X_{\mathrm{inf}}}^p)^{\frac{1}{p}}. \tag{3.2}$$

Let us use (3.2) first to show that $\|\cdot\|_{X_{inf}}$ is a quasi-norm. Indeed if $x, y \in c_{00}$ then

$$\|x + y\|_{X_{inf}} \le 2\|\max(|x|, |y|)\|_{X_{inf}} \le 2^{a+1} C (\|x\|_{X_{inf}} + \|y\|_{X_{inf}}).$$

Next note that (3.2) implies

$$\|x_1 + \cdots + x_k\|_{X_{inf}} \leq Ck^{a+\frac{1}{p}} \max_{1 \leq j \leq k} \|x_j\|_{X_{inf}}.$$

From this it follows easily that if $a + \frac{1}{p} < \frac{1}{r} < 1$ we have

$$\|x_1 + \cdots + x_k\|_{X_{inf}} \leq C_r (\sum_{j=1}^{k} \|x_j\|^r)^{\frac{1}{r}}$$

for disjoint $x_1, \cdots, x_k$. Thus we have an upper r-estimate for X_{inf}.

It is trivial to show X_{inf} has a lower q-estimate. Now by [6, Theorem 4.1] (a simpler proof is given in [7, Theorem 3.2]) it follows that X_{inf} is *lattice-convex* and this means that an upper r-estimate implies (lattice) s-convexity for all $s < r$ (Theorem 2.2 of [6]). Hence X_{inf} is r-convex for every r with $a + \frac{1}{p} < \frac{1}{r}$. In particular 1-convexity implies the quasi-norm is equivalent to a norm. In fact X_{inf}^* is a reflexive Banach space.

Now it is obvious that $X_{inf} \subset (X_{sup}^*)^*$ and $X_{sup}^* \subset (X_{inf})^*$. Hence it follows easily that $(X_{inf})^* = X_{sup}^*$. $\square$

Remark: We can apply the above result to the case of the weighted ℓ_p-space X, with $1 < p < \infty$ defined by the norm

$$\|\xi\|_X = (\sum_{n=1}^{\infty} |\xi_n|^p w_n)^{\frac{1}{p}}$$

where (w_n) is an increasing sequence satisfying an estimate of the form

$$w_{kn} \leq Ck^a w_n$$

where $a < p - 1$. The X_{inf} is defined by the quasi-norm

$$\|\xi\|_{X_{inf}} = (\sum_{n=1}^{\infty} (\xi_n^*)^p w_n^p)^{\frac{1}{p}}$$

where (ξ_n^*) is the decreasing rearrangement of $(|\xi_n|)$. In this case X_{sup} is the Lorentz space $d((w_n)^{-q/p}, q)$.

This result can be rephrased. If (v_n) is a positive decreasing sequence satisfying an estimate $v_n \leq Ck^b v_{kn}$ where $b < 1$ then $d((v_n), p)^*$ can be identified with the space of all sequences (ξ_n)

so that

$$(\sum_{n=1}^{\infty}(\xi_n^*)^q v_n^{-q/p})^{\frac{1}{q}} < \infty.$$

This result is a special case of results of Reisner [15].

Proposition VIII.a.8 of [3] states that the decreasing rearrangement operator D is a bounded non-linear operator on the original Tsirelson space T. This result then immediately carries over to the 2-convexification of T which is our convexified Tsirelson space T^2. By Proposition 2.8 we have that Theorem 3.4 holds in this case. We summarize this in the following result:

Proposition 3.5 *There is a constant $K \geq 1$ so that for any $x = \sum_n a_n t_n \in T^2$ we have*

$$inf_{\sigma\in\Pi}\|\sum_n a_{\sigma(n)}t_n\| \leq \|Dx\| \leq K inf_{\sigma\in\Pi}\|\sum_n a_{\sigma(n)}t_n\|. \tag{3.3}$$

Moreover, there is a norm $\|\cdot\|_s$ on the set of vectors for which $\|Dx\| < \infty$ satisfying

$$\frac{1}{K}\|x\|_s \leq \|Dx\| \leq K\|x\|_s. \tag{3.4}$$

Note that our operator D does not satisfy a triangle inequality, but does with the constant K on the sum side of the triangle inequality.

Definition 3.6 *The symmetric convexified Tsirelson space is the Banach space $S(T^2)$ of vectors for which $\|x\|_s < \infty$ with natural unit vector basis (t_n^s). By Theorem 3.4 this is a reflexive Banach space whose dual space is $S[(T^2)^*]$.*

It is known [3] that every infinite dimensional subspace of $S(T^2))$ contains a subspace which embeds into T^2. In particular $S(T^2)$ is a Banach space with a natural symmetric basis which has no subspaces isomorphic to c_0 or ℓ_p for $1 \leq p < \infty$. Also T^2 embeds into $S(T^2)$. Since the unit vector basis of ℓ_2 uniformly dominates all block bases of (t_n) in T^2, it follows that the unit vector basis of $S(T^2)$ is also dominated by the unit vector basis of ℓ_2.

Proposition 3.7 *The space $S(T^2)$ fails property upper (H) (even for disjointly supported elements) and fails property lower (H_2). Hence $S(T^2)$ is not of weak type 2 and not of cotype 2.*

Proof: First we check property lower (H_2). Since (t_n^s) is symmetric and is dominated by the unit vector basis of ℓ_2, it follows that if this family had subsets dominating the unit vector basis of ℓ_2, then (t_n^s) would be equivalent to the unit vector basis of ℓ_2 which is impossible.

For property upper (H), fix $M > 1$ and choose a decreasing sequence of non-zero scalars $(a_i)_{i=1}^n$ whose ℓ_2 norm is $> M$ but $\|\sum_i a_i t_i\|_{T^2} = 1$. This can be done by a modification of the constructions of [3, Chapter IV]. Now let $(x_j)_{j=1}^n$ be a sequence of disjoint vectors in $S(T^2)$ which have this set of $a_i's$ as coefficients. So $\|x_i\|_{S(T^2)} = 1$ for every $i = 1, 2, \cdots, n$. But to norm $\sum_i x_i$ in $S(T^2)$, we have to arrange all the coefficients in decreasing order and take the norm in T^2. Since these vectors are disjoint, at least half of them, say $(x_i)_{i \in I}$, will have all of their support after $t_{n/2}$. That is, we have n/2 vectors in T^2 which are disjoint and have their supports after $t_{n/2}$. Hence

$$\begin{aligned}\|\sum_{i=1}^n x_i\|_{ST^2} &\geq K^{-1}\|D\sum_{i=1}^n x_i\|_{T^2} \geq K^{-1}(\sum_{i\in I}\|x_i\|_{T^2}^2)^{1/2} \\ &\geq K^{-2}(\sum_{i\in I}\|x_i\|_{ST^2}^2)^{1/2} \geq K^{-2}M(\frac{n}{2})^{1/2}.\end{aligned}$$

Since M was arbitrarily large, it follows that $S(T^2)$ fails upper (H) - for disjoint elements.
$\square$

We shall now need a result essentially due to S. Kwapien. In the form we present it is due to W.B. Johnson and it appeared in [8]

Proposition 3.8 *There is a function*

$$N(k, \epsilon) = \left[\frac{2k^2}{\epsilon}\right]^k$$

such that for any fixed $0 < \epsilon < 1$, every order complete Banach Lattice L, and every k-dimensional subspace F of L, there are $N = N(k, \epsilon)$ disjoint elements $(x_j)_{j=1}^N$ in L and a linear operator $V : F \to X = span(x_j)$ such that for all $x \in X$ we have

$$\|Vx - x\| \leq \epsilon\|x\|.$$

Proposition 3.9 *There is a constant $K > 1$ so that for every subspace E of $S(T^2)$ of dimension n, we have for all $i \in \mathbb{N}$ for which $log_{i-1}n$ exists,*

$$d(E, \ell_2^n) \leq K^i log_{i-2}n.$$

Proof: By giving up one level of logs we may assume by Proposition 3.8 that we are working with a normalized disjointly supported sequence of vectors $(x_j)_{j=1}^n$ in $S(T^2)$. Now there is a disjoint set of permutations y_j of the x_j so that

$$\begin{aligned} \|\sum_{j=1}^n a_j x_j\|_{ST^2} &\geq \frac{1}{K}\|\sum_{j=1}^n a_j y_j\|_{T^2} \\ \geq \frac{1}{K}\|\sum_{j=1}^n a_j t_j\|_{T^2} &\geq \frac{1}{K^{i+1}(\log_i n)}\left(\sum_{j=1}^n |a_j|^2\right)^{1/2}. \end{aligned}$$

Also, let $Dx_j = z_j$ and

$$w_j = \sum_k z_j(k) t_{n(k-1)+j},$$

By Proposition 2.8 we have

$$\begin{aligned} \|D\sum_{j=1}^n a_j x_j\|_{ST^2} &\leq K\|\sum_{j=1}^n a_j w_j\|_{T^2} \leq 2K\left(\sum_{j=1}^n |a_j|^2 \|w_j\|_{T^2}^2\right)^{1/2} \\ &\leq 2K\left(\sum_{j=1}^n |a_j|^2 [2K^i(\log_i n)]^2\right)^{1/2} \leq 4K^{i+1}(\log_i n)\left(\sum_{j=1}^n |a_j|^2\right)^{1/2}, \end{aligned}$$

and hence

$$d(E, \ell_2^n) \leq 4K^{2(i+1)}(\log_i n)^2 \leq K^i(\log_{i-1} n).$$

The $\log_{i-2}n$ in the statement of the theorem comes from the fact that we first applied Proposition 3.8. □

Corollary 3.10 *The space $S(T^2)$ is of type p for all $1 \leq p < 2$ and of cotype q for all $2 < q$.*

Before we go on, we need a criterion for a Banach space to be of weak cotype 2. We shall say that a Banach space X has *property* (P) if there is a constant K so that whenever

$\{x_1, x_2, \ldots, x_n\} \subseteq X$ is a finite set with $\max_{1\le j\le n} |t_j| \le \| \sum_{j=1}^n t_j x_j \|$ for all $(t_j) \subseteq \mathbb{R}$, then

$$\sqrt{n} \le K\Big(\int \| \sum_{j=1}^n g_j(t) x_j \|^2 d\mu(t) \Big)^{\frac{1}{2}} \tag{3.5}$$

It was proved by Pisier [14, Proposition 10.8] that if X is of weak cotype 2, then it has property (P). It turns out that (P) characterizes weak cotype 2 spaces. This fact might be known to specialists but we shall give a short proof here:

Theorem 3.11 *If X has property (P), then it is of weak cotype 2.*

Proof: Let $E \subseteq X$ be a finite dimensional subspace, say $\dim(E) = 2n$. By a result of Bourgain and Szarek [2, Theorem 2] there is a universal constant C and $\{x_1, x_2, \ldots, x_n\} \subseteq X$ so that for all $(t_j) \subseteq \mathbb{R}$ we have

$$\max_{1\le j\le n} |t_j| \le \| \sum_{j=1}^n t_j x_j \| \le C\Big(\sum_{j=1}^n |t_j|^2 \Big)^{\frac{1}{2}} \tag{3.6}$$

Using property (P) we get that

$$\sqrt{n} \le K\Big(\int \| \sum_{j=1}^n g_j(t) x_j \|^2 d\mu(t) \Big)^{\frac{1}{2}} \tag{3.7}$$

where K is the constant of property (P). Now, (3.7) and the right inequality of (3.6) give together with one of main results of [4, Theorem 2.6] (see also [16, pages 25 and 81]) that there is a universal constant η such that if $k \le \eta K^{-2} C^{-2} n$, then there is a k-dimensinal subspace $F \subseteq [x_j]$ with $d(F, l_2^k) \le 2$. From [14, Theorem 10.2] it now follows that X is of weak cotype 2. $\square$

We shall say that a sequence $(x_j)_{j=1}^n$ in a Banach space X is 1-separated if $\|x_i - x_j\| \ge 1$ for all $1 \le i, j \le n$, $i \ne j$. It follows immediately from Theorem 3.11 that if every 1-separated sequence in X satisfies (3.5), then X is of weak cotype 2.

We are now ready to prove that the symmetric convexified Tsirelson space is a weak cotype 2 space with a symmetric basis which is not of cotype 2. Hence its dual space is a symmetric space which is of weak type 2 but fails to be of type 2.

Theorem 3.12 *The space $S(T^2)$ is a weak cotype 2 space.*

Proof: Let $(x_j)_{j=1}^n$ be a 1-separated sequence in $S(T^2)$. Without loss of generality we may assume that for all $1 \leq i \leq n$ we have $\|x_i\|_{S(T^2)} \geq 1$. We wish to show that (3.5) holds. If K is a constant which satisfies (1.5) for both T^2 and $S(T^2)$ and (3.4), then by definition we can find a $\sigma \in \Pi$ so that:

$$\|(\sum_{j=1}^n |S_\sigma x_j|^2)^{\frac{1}{2}}\|_{T^2} = \|S_\sigma(\sum_{j=1}^n |x_j|^2)\|_{T^2} \leq K\|(\sum_{j=1}^n |x_j|^2)^{\frac{1}{2}}\|_{S(T^2)} \tag{3.8}$$

Since S_σ is an isometry on $S(T^2)$, we can without loss of generality assume that actually $x_j = S_\sigma x_j$ for all $1 \leq j \leq n$.

Put $k = \log\log n$ and let P_k be the natural projection of T^2 onto the span of $(t_j)_{j=1}^k$. We now examine two cases.

Case I: There is a subset $I \subset \{1, 2, \cdots, n\}$ with $|I| \geq \frac{n}{2}$ so that $\|P_k x_j\|_{\ell_2} \geq \log k$ for all $j \in I$.

Since $(t_j)_{j=1}^k$ is $K\log k$-isomorphic to a Hilbert space by Proposition 2.7, we get using (1.5) and (3.8)

$$\begin{aligned}
\Big(\int \|\sum_{j=1}^n g_j(t)x_j\|_{S(T^2)}^2 d\mu(t)\Big)^{\frac{1}{2}} &\geq \frac{1}{K}\|(\sum_{j=1}^n |x_j|^2)^{1/2}\|_{S(T^2)} \geq \frac{1}{K^2}\|(\sum_{j=1}^n |x_j|^2)^{1/2}\|_{T^2} \\
&\geq \frac{1}{K^2}\|(\sum_{j=1}^n |P_k x_j|^2)^{1/2}\|_{T^2} \geq \frac{1}{(\log k)K^3}\|(\sum_{j\in I} |P_k x_j|^2)^{1/2}\|_{\ell^2} \\
&= \frac{1}{(\log k)K^3}(\sum_{j\in I} \|P_k x_j\|^2)^{\frac{1}{2}} \geq \frac{1}{K^3\sqrt{2}}\sqrt{n}
\end{aligned} \tag{3.9}$$

Case II: There is a subset $I \subset \{1, 2, \cdots, n\}$ with $|I| \geq \frac{n}{2}$ so that $\|P_k x_j\|_{\ell_2} \leq \log k$ for all $j \in I$.

In this case we make the following claim:

Claim: There is a subset $J \subset I$ with $|J| \geq \frac{n}{4}$, so that for all $j \in J$,

$$\|(I - P_k)x_j\|_{T^2} \geq \frac{1}{8K}.$$

If not, there is a set J as above with

$$\|(I - P_k)x_j\|_{T^2} \leq \frac{1}{8K}.$$

By a volume of the ball argument (see e.g. [4, Lemma 2.4]) the cardinality of a set of points which are $\frac{1}{4K}$ apart in a ball of radius $\log k$ in k-dimensional Hilbert space is at most $(1+8K\log k)^k$ which by our choice of k is less than or equal to $\frac{n}{4}$ (at least for large n). Hence there exist $i, j \in J$, $i \neq j$ so that

$$\|P_k(x_i - x_j)\|_{\ell^2} \leq \frac{1}{4K}.$$

Now we compute

$$\begin{aligned}
\|x_i - x_j\|_{S(T^2)} &\leq K\|x_i - x_j\|_{T^2} \leq K\|P_k(x_i - x_j)\|_{T^2} + K\|(I-P_k)x_i\|_{T^2} + K\|(I-P_k)x_j\|_{T^2} \\
&\leq K\|P_k(x_i - x_j)\|_{\ell_2} + K\frac{1}{8K} + K\frac{1}{8K} \leq K\frac{1}{4K} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

This contradicts our 1-separation assumption. So the claim holds.

Now by the claim, the beginning of the proof, (1.5) and Proposition 2.6 we get

$$\begin{aligned}
\Big(\int \Big\|\sum_{j=1}^n g_j(t)x_j\Big\|_{S(T^2)}^2 d\mu(t)\Big)^{\frac{1}{2}} &\geq \frac{1}{K^2}\|(\sum_{j=1}^n |x_j|^2)^{1/2}\|_{T^2} \\
&\geq \frac{1}{K^2}\|(I-P_k)(\sum_{j\in J}|x_j|^2)^{1/2}\|_{T^2} \\
&\geq \frac{1}{K^3}\Big(\int \Big\|\sum_{j=1}^n g_j(t)(I-P_k)x_j\Big\|_{T^2}^2 d\mu(t)\Big)^{\frac{1}{2}} \\
&\geq \geq \frac{1}{K^5}\left(\sum_{j\in J}\|(I-P_k)x_j\|_{T^2}^2\right)^{1/2} \\
&\geq \frac{1}{K^5}\left(\sum_{j\in J}(\frac{1}{8K})^2\right)^{1/2} \geq \frac{|J|^{1/2}}{8K^6} \geq \frac{\sqrt{n}}{16K^6}
\end{aligned} \tag{3.10}$$

This completes the proof. ☐

As a corollary we obtain:

Corollary 3.13 *Even for Banach lattices property upper H and the weak type 2 property do not imply the upper H_2 property. Similarly, property lower H and the weak cotype 2 property do not imply the lower H_2 property.*

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